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Four-dimensional Walker metrics and symplectic structures

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Dedicated to Professor K. Sekigawa on his 60th birthday

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Abstract

It is interesting to recognize that a nonflat indefinite Kähler–Einstein metric on a torus constructed by Petean is an example of four-dimensional Walker metrics. We show that generic orientable Walker metrics in dimension four admit a pair of an almost complex structure and an opposite almost complex structure, and further consider their integrability, the existence condition of symplectic structures for these metrics. We shall see that such a family of Walker 4-manifolds contains a class of indefinite Kähler–Einstein 4-manifolds to which Petean's metric belongs, examples of indefinite Hermitian 4-manifolds, and also examples of indefinite almost Kähler 4-manifolds.

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1. Introduction

It is known that the normal form of metrics on a manifold with a field of parallel null planes of arbitrary dimension has determined by Walker [7,8]. Such a normal form in dimension four with a field of parallel null 2-planes is the lowest dimensional example. Note that four-dimensional Walker metrics are neutral, i.e., of indefinite signature $(+\,+\,-\,-)$. It is also known that the existence condition for an indefinite metric of signature $(+\,+\,-\,-)$,

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with the structure group $SO_0(2,2)$, on a 4-manifold is equivalent to the existence of a nonsingular field of oriented tangent 2-planes, and moreover to the existence of a pair of an almost complex structure and an opposite almost complex structure on the manifold [3–5].

From a fact that a nonflat indefinite Kähler–Einstein metric on a torus constructed by Petean [6] is an example of Walker metrics of certain restricted type, it is interesting to study two kinds of almost complex structures associated with the four-dimensional Walker metrics.

In the present note, we shall focus our attention to the Walker metrics of certain restricted type, and among them we find a class of indefinite Kähler-Einstein metrics which contains Petean's example, a class of indefinite Hermitian metrics, a class of indefinite almost Kähler metrics, and others.

2. Walker metrics in dimension four

A 4-dimensional Walker manifold is a triple (M, g, D) of a four-manifold M, together with an indefinite metric g and a nonsingular two-dimensional distribution D such that D is parallel and null with respect to g. Such a 4-manifold M is the lowest dimensional example among generic Walker manifolds.

From Walker's theorem ([7, Theorem 1 and Section 6, Case 1]), we see that a canonical form of g is given by

$$g = [g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{bmatrix}, \tag{1}$$

where a, b and c are functions of the coordinates (x^1, x^2, x^3, x^4). Thus g is of signature $(+\,+\,-\,-)$, or neutral. The parallel null two-plane D is spanned locally by $\{\partial/\partial x^1, \partial/\partial x^2\}$.

From now on we shall restrict our attention to an orientable Walker 4-manifold, i.e., the structure group of the tangent bundle is the identity component $SO_0(2,2)$ of $O(2,2)$. A fundamental fact for our present issue is the following (cf. [3–5]):

Theorem 1. *An orientable Walker 4-manifold (M, g, D) admits a pair of an almost complex structure J and an opposite almost complex structure J', such that J and J' commute with each other.*

3. Nonflat indefinite Kähler metrics on tori as Walker metrics

On a torus $T = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2$, Petean [6] defined the following Kähler form

$$\Omega = \tfrac{1}{2}\mathrm{i}(\mathrm{d}z \wedge \mathrm{d}\bar{w} + \mathrm{d}w \wedge \mathrm{d}\bar{z} + f(w)\,\mathrm{d}w \wedge \mathrm{d}\bar{w}), \tag{2}$$

where $f(w)$ is a smooth positive function on $\mathbb{C}/\Lambda_2$ (see Remark below). This gives a nonflat indefinite Kähler–Einstein metric

$$g = \mathrm{d}z\,\mathrm{d}\bar{w} + \mathrm{d}w\,\mathrm{d}\bar{z} + f(w)\,\mathrm{d}w\,\mathrm{d}\bar{w}. \tag{3}$$

Put $(z, w) = (x_1 + \mathrm{i}x_2, x_3 + \mathrm{i}x_4)$, then in terms of real coordinates Ω and g are written respectively as

$$\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^4 - \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f(x_3, x_4)\,\mathrm{d}x^3 \wedge \mathrm{d}x^4, \tag{4}$$

$$g = 2\,\mathrm{d}x^1\,\mathrm{d}x^3 + 2\,\mathrm{d}x^2\,\mathrm{d}x^4 + f(x^3, x^4)\{(\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2\}. \tag{5}$$

Therefore, the metric tensor $[g_{ij}]$ takes the form

$$g = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & f & 0 \\ 0 & 1 & 0 & f \end{bmatrix}. \tag{6}$$

It turns out that this metric g is a special case of the Walker metric (1) such that

$$a = b = f(x^3, x^4), \qquad c = 0. \tag{7}$$

Remark. The coordinates z and w in [6, p. 233] are interchanged here, in order for the metric g in real coordinates coincides with the canonical form (1).

4. Two kinds of almost complex structures

A natural way to construct a pair of an almost complex structure J and an opposite almost complex structure J' on a neutral 4-manifold is as follows: choose a local orthonormal basis $\{\boldsymbol{e}_i\}(i = 1, \ldots, 4)$ so that with respect to the basis the neutral metric becomes the standard form

$$g = [g(\boldsymbol{e}_i, \boldsymbol{e}_j)] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{8}$$

and then define J and J' by

$$J\boldsymbol{e}_1 = \boldsymbol{e}_2, \qquad J\boldsymbol{e}_2 = -\boldsymbol{e}_1, \qquad J\boldsymbol{e}_3 = \boldsymbol{e}_4, \qquad J\boldsymbol{e}_4 = -\boldsymbol{e}_3, \tag{9}$$

$$J'\boldsymbol{e}_1 = \boldsymbol{e}_2, \qquad J'\boldsymbol{e}_2 = -\boldsymbol{e}_1, \qquad J'\boldsymbol{e}_3 = -\boldsymbol{e}_4, \qquad J'\boldsymbol{e}_4 = \boldsymbol{e}_3. \tag{10}$$

It is known that $Q = -JJ' = -J'J$ defines an almost product structure satisfying $Q^2 = 1$.

Associated with these structures J, J' and g, we have two kinds of Kähler forms on the 4-manifold as follows:

$$\Omega_g(X, Y) = g(JX, Y), \qquad \Omega'_g(X, Y) = g(J'X, Y). \tag{11}$$

In terms of the local orthonormal basis $\{\boldsymbol{e}^i\}$ of one-forms, these Kähler forms are written explicitly as follows:

$$\Omega_g = \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 - \boldsymbol{e}^3 \wedge \boldsymbol{e}^4, \qquad \Omega'_g = \boldsymbol{e}^1 \wedge \boldsymbol{e}^2 + \boldsymbol{e}^3 \wedge \boldsymbol{e}^4. \tag{12}$$

We must note that $\Omega_g \wedge \Omega_g = -\Omega'_g \wedge \Omega'_g = -2\boldsymbol{e}^1 \wedge \boldsymbol{e}^2 \wedge \boldsymbol{e}^3 \wedge \boldsymbol{e}^4$, and therefore Ω'_g has the manifold orientation, but Ω_g the reversed orientation as the preferred one.

In the present note, we shall consider the Walker 4-manifolds of certain restricted type, i.e., a case of $c = 0$, which includes Petean's nonflat indefinite Kähler metric on a torus.

5. J, J', Ω_g and Ω'_g in the case of $c = 0$

We consider the Walker metrics with $c = 0$ as follows:

$$[g_{ij}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{bmatrix}, \tag{13}$$

where a and b are functions of (x^1, x^2, x^3, x^4). In this case, we find a local orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ as follows:

$$\begin{aligned}
\boldsymbol{e}_1 &= \frac{1}{\sqrt[4]{a^2+4}} \left\{ \frac{1}{2}\left(\sqrt{a^2+4} - a\right) \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3} \right\}, \\
\boldsymbol{e}_2 &= \frac{1}{\sqrt[4]{b^2+4}} \left\{ \frac{1}{2}\left(\sqrt{b^2+4} - b\right) \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} \right\}, \\
\boldsymbol{e}_3 &= \frac{1}{\sqrt[4]{a^2+4}} \left\{ -\frac{1}{2}\left(\sqrt{a^2+4} + a\right) \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3} \right\}, \\
\boldsymbol{e}_4 &= \frac{1}{\sqrt[4]{b^2+4}} \left\{ -\frac{1}{2}\left(\sqrt{b^2+4} + b\right) \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4} \right\}.
\end{aligned} \tag{14}$$

Relative to the above basis, the metric (13) becomes the standard form as in (8).

The almost complex structure J defined by (9) acts explicitly on the coordinate basis as follows:

$$\begin{aligned}
&J\frac{\partial}{\partial x^1} = K\frac{\partial}{\partial x^2}, \qquad J\frac{\partial}{\partial x^2} = -\frac{1}{K}\frac{\partial}{\partial x^1}, \\
&J\frac{\partial}{\partial x^3} = \frac{1}{2}\left(Ka - \frac{b}{K}\right)\frac{\partial}{\partial x^2} + \frac{1}{K}\frac{\partial}{\partial x^4}, \\
&J\frac{\partial}{\partial x^4} = \frac{1}{2}\left(Ka - \frac{b}{K}\right)\frac{\partial}{\partial x^1} - K\frac{\partial}{\partial x^3},
\end{aligned} \tag{15}$$

where we have put

$$K = \sqrt[4]{\frac{b^2+4}{a^2+4}}. \tag{16}$$

Similarly, the opposite almost complex structure J' defined by (10) is also a linear operator on T_pM as

$$J'\frac{\partial}{\partial x^1}=\frac{1}{H}\left(-b\frac{\partial}{\partial x^2}+2\frac{\partial}{\partial x^4}\right),\qquad J'\frac{\partial}{\partial x^2}=\frac{1}{H}\left(a\frac{\partial}{\partial x^1}-2\frac{\partial}{\partial x^3}\right),$$
$$J'\frac{\partial}{\partial x^3}=\frac{1}{2}\left(H-\frac{ab}{H}\right)\frac{\partial}{\partial x^2}+\frac{a}{H}\frac{\partial}{\partial x^4},\qquad J'\frac{\partial}{\partial x^4}=-\frac{1}{2}\left(H-\frac{ab}{H}\right)\frac{\partial}{\partial x^1}-\frac{b}{H}\frac{\partial}{\partial x^3}, \tag{17}$$

where we have put

$$H=\sqrt[4]{(a^2+4)(b^2+4)}. \tag{18}$$

For the Walker metric (13) with $c=0$, the dual basis $\{\boldsymbol{e}^1,\boldsymbol{e}^2,\boldsymbol{e}^3,\boldsymbol{e}^4\}$ of 1-forms to the basis (14) of vectors is given by

$$\begin{aligned}
\boldsymbol{e}^1&=\frac{1}{\sqrt[4]{a^2+4}}\left\{\mathrm{d}x^1+\frac{1}{2}\left(\sqrt{a^2+4}+a\right)\mathrm{d}x^3\right\},\\
\boldsymbol{e}^2&=\frac{1}{\sqrt[4]{b^2+4}}\left\{\mathrm{d}x^2+\frac{1}{2}\left(\sqrt{b^2+4}+b\right)\mathrm{d}x^4\right\},\\
\boldsymbol{e}^3&=-\frac{1}{\sqrt[4]{a^2+4}}\left\{\mathrm{d}x^1-\frac{1}{2}\left(\sqrt{a^2+4}-a\right)\mathrm{d}x^3\right\},\\
\boldsymbol{e}^4&=-\frac{1}{\sqrt[4]{b^2+4}}\left\{\mathrm{d}x^2-\frac{1}{2}\left(\sqrt{b^2+4}-b\right)\mathrm{d}x^4\right\}.
\end{aligned} \tag{19}$$

Therefore, the basis $\{\boldsymbol{e}^i\wedge\boldsymbol{e}^j\}$ for 2-forms is written as follows:

$$\begin{aligned}
\boldsymbol{e}^1\wedge\boldsymbol{e}^2&=\frac{1}{H}\left\{\mathrm{d}x^1\wedge\mathrm{d}x^2+\frac{1}{2}\left(\sqrt{b^2+4}+b\right)\mathrm{d}x^1\wedge\mathrm{d}x^4\right.\\
&\quad+\frac{1}{4}\left(\sqrt{a^2+4}+a\right)\left(\sqrt{b^2+4}+b\right)\mathrm{d}x^3\wedge\mathrm{d}x^4\\
&\quad\left.-\frac{1}{2}\left(\sqrt{a^2+4}+a\right)\mathrm{d}x^2\wedge\mathrm{d}x^3\right\},\\
\boldsymbol{e}^1\wedge\boldsymbol{e}^3&=\mathrm{d}x^1\wedge\mathrm{d}x^3,\\
\boldsymbol{e}^1\wedge\boldsymbol{e}^4&=-\frac{1}{H}\left\{\mathrm{d}x^1\wedge\mathrm{d}x^2-\frac{1}{2}\left(\sqrt{b^2+4}-b\right)\mathrm{d}x^1\wedge\mathrm{d}x^4\right.\\
&\quad-\frac{1}{4}\left(\sqrt{a^2+4}+a\right)\left(\sqrt{b^2+4}-b\right)\mathrm{d}x^3\wedge\mathrm{d}x^4\\
&\quad\left.-\frac{1}{2}\left(\sqrt{a^2+4}+a\right)\mathrm{d}x^2\wedge\mathrm{d}x^3\right\},
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{e}^3 \wedge \boldsymbol{e}^4 &= \frac{1}{H}\left\{\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \frac{1}{2}\left(\sqrt{b^2+4}-b\right)\mathrm{d}x^1 \wedge \mathrm{d}x^4\right.\\
&\quad + \frac{1}{4}\left(\sqrt{a^2+4}-a\right)\left(\sqrt{b^2+4}-b\right)\mathrm{d}x^3 \wedge \mathrm{d}x^4\\
&\quad \left. + \frac{1}{2}\left(\sqrt{a^2+4}-a\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3\right\},\\
\boldsymbol{e}^4 \wedge \boldsymbol{e}^2 &= \mathrm{d}x^4 \wedge \mathrm{d}x^2,\\
\boldsymbol{e}^2 \wedge \boldsymbol{e}^3 &= \frac{1}{H}\left\{\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \frac{1}{2}\left(\sqrt{b^2+4}+b\right)\mathrm{d}x^1 \wedge \mathrm{d}x^4\right.\\
&\quad - \frac{1}{4}\left(\sqrt{a^2+4}-a\right)\left(\sqrt{b^2+4}+b\right)\mathrm{d}x^3 \wedge \mathrm{d}x^4\\
&\quad \left. + \frac{1}{2}\left(\sqrt{a^2+4}-a\right)\mathrm{d}x^2 \wedge \mathrm{d}x^3\right\}.
\end{aligned}
\tag{20}
$$

Then, two kinds of Kähler forms in (12) are explicitly written in terms of the coordinate basis as follows:

$$
\Omega_g = K\,\mathrm{d}x^1 \wedge \mathrm{d}x^4 - \frac{1}{K}\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{1}{2}\left(aK + \frac{b}{K}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^4,
\tag{21}
$$

$$
\begin{aligned}
\Omega'_g &= \frac{1}{H}(2\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + b\,\mathrm{d}x^1 \wedge \mathrm{d}x^4 - a\,\mathrm{d}x^2 \wedge \mathrm{d}x^3)\\
&\quad + \frac{1}{2}\left(\frac{ab}{H} + aK + \frac{b}{K}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^4.
\end{aligned}
\tag{22}
$$

6. Symplectic structures in the case of $c = 0$

At this stage, we shall consider if M admits a symplectic structure ($\mathrm{d}\Omega_g = 0$), and the integrability of J. Concerning symplectic structure, we have the following theorem.

Theorem 2. *The Kähler form Ω_g is a symplectic form* ($\mathrm{d}\Omega_g = 0$) *if the following partial differential equations hold*

$$
\begin{aligned}
&\frac{\partial K}{\partial x^1} = 0, \qquad \frac{\partial K}{\partial x^2} = 0, \qquad K^2\frac{\partial a}{\partial x^1} + \frac{\partial b}{\partial x^1} - 2K\frac{\partial K}{\partial x^3} = 0,\\
&K^2\frac{\partial a}{\partial x^2} + \frac{\partial b}{\partial x^2} + \frac{2}{K}\frac{\partial K}{\partial x^4} = 0.
\end{aligned}
\tag{23}
$$

In this note, we will not try to find the general solutions for the PDE's in the above theorem, but consider some restricted cases. From the theorem, we immediately see that if K is constant, the first two conditions become trivial, and that the last two conditions reduce to the following equation:

$$
\frac{\partial}{\partial x^1}(K^2 a + b) = 0, \qquad \frac{\partial}{\partial x^2}(K^2 a + b) = 0.
\tag{24}
$$

It is clear that if a and b are constant, then the above two conditions hold too. Therefore, it is clear that Ω_g in (21) and Ω'_g in (22) are both symplectic structures, and that J in (15) and J' in (17) are both integrable. This is a trivial case of the flat Walker metric.

There is a situation such that K is constant, but a and b are not constant. Such a case occurs if b^2+4 is a constant multiple of a^2+4 for $a = a(x^1, x^2, x^3, x^4)$ and $b = b(x^1, x^2, x^3, x^4)$. In this case, only the first two conditions of Theorem 2 hold since K is constant, but the conditions (24) may not be satisfied by generic a and b.

In what follows, we consider the conditions (24) and integrability of J in some cases.

Case I (K is constant, and $a = a(x^3, x^4)$ and $b = b(x^3, x^4)$). In this case, the first two conditions of Theorem 2 hold, and conditions (24) hold too, since a and b do not depend on x^1 and x^2. Some algebra show that the Nijenhuis tensor of J in (15) vanishes, and therefore J is integrable.

Remark (Einstein condition). If $a = a(x^3, x^4)$ and $b = b(x^3, x^4)$, then the metric (13) is of Einstein without any restriction on K, i.e., for K either being constant or nonconstant.

We thus have the following theorem.

Theorem 3. *Suppose that b^2+4 is a constant multiple of a^2+4* for $a = a(x^1, x^2, x^3, x^4)$ *and $b = b(x^1, x^2, x^3, x^4)$. If a depends only on (x^3, x^4), then the Walker four-manifold (M, g, D) admits a symplectic structure, an integrable almost complex structure J, and g is of Einstein. Thus, the Walker 4-manifolds of Case I admit an indefinite Kähler-Einstein structure.*

If a and b are functions of (x^3, x^4) such that $b^2+4 = C^4(a^2+4)$ (C: constant), then the 2-form

$$\Omega_g = C\,\mathrm{d}x^1 \wedge \mathrm{d}x^4 - \frac{1}{C}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \frac{1}{2}\left(Ca(x^3, x^4) + \frac{b(x^3, x^4)}{C}\right)\mathrm{d}x^3 \wedge \mathrm{d}x^4 \qquad (25)$$

is an example of the symplectic forms of this type.

The case of $K = 1$ occurs if $a = b$ or $b = -a$ for $a = a(x^1, x^2, x^3, x^4)$ and $b = b(x^1, x^2, x^3, x^4)$.

Case II ($a(x^1, x^2, x^3, x^4) = b(x^1, x^2, x^3, x^4)$). In this case, Ω_g and Ω'_g are not in general symplectic for generic a ($= b$). This case is, however, important due to the following fact.

Proposition 4. *Suppose that $a(x^1, x^2, x^3, x^4) = b(x^1, x^2, x^3, x^4)$* and $c = 0$. *Then the Walker four-manifold (M, g, D) admits a complex structure (integrable almost complex structure).*

Proof. Action of J in (15) reduces to the following equation:

$$J\frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^2}, \qquad J\frac{\partial}{\partial x^2} = -\frac{\partial}{\partial x^1}, \qquad J\frac{\partial}{\partial x^3} = \frac{\partial}{\partial x^4}, \qquad J\frac{\partial}{\partial x^4} = -\frac{\partial}{\partial x^3}, \qquad (26)$$

which implies that J is integrable. □

Since action of J is standard as in (26), the Walker metric can be written in terms of complex coordinates (z, w) $(= (x^1 + \mathrm{i}x^2, x^3 + \mathrm{i}x^4))$ as follows:

$$\begin{aligned} g &= 2\,\mathrm{d}x^1\,\mathrm{d}x^3 + 2\,\mathrm{d}x^2\,\mathrm{d}x^4 + a(x^1, x^2, x^3, x^4)\{(\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2\} \\ &= \mathrm{d}z\,\mathrm{d}\bar{w} + \mathrm{d}w\,\mathrm{d}\bar{z} + a(z, w)\,\mathrm{d}w\,\mathrm{d}\bar{w}, \end{aligned} \tag{27}$$

where a is a smooth function of z and w. There exists such an indefinite Hermitian metric on the four-space $M = \mathbb{R}^4 = \mathbb{C}^2$, which is an example of noncompact Walker four-manifolds. There exists also a compact indefinite Hermitian 4-manifold on a complex torus $T = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2$, with a metric of the type.

Case III $(a(x^3, x^4) = b(x^3, x^4))$. This is a common case of Cases I and II, i.e., Case III = Case I $\cap$ Case II. Therefore, the Walker 4-manifold of the present class admits an integrable almost complex structure J and a symplectic structure.

Then we have the following proposition.

Proposition 5. *Suppose that* $a(x^3, x^4) = b(x^3, x^4)$ and $c = 0$. *Then the Walker 4-manifold* (M, g, D) *admits an indefinite Kähler–Einstein structure*.

The symplectic structure in (25) now becomes

$$\begin{aligned} \Omega_g &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 - \mathrm{d}x^2 \wedge \mathrm{d}x^3 + a(x^3, x^4)\,\mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ &= \tfrac{1}{2}\mathrm{i}(\mathrm{d}z \wedge \mathrm{d}\bar{w} + \mathrm{d}w \wedge \mathrm{d}\bar{z} + a(w)\,\mathrm{d}w \wedge \mathrm{d}\bar{w}). \end{aligned} \tag{28}$$

We see that the metric in (27) yields

$$\begin{aligned} g &= 2\mathrm{d}x^1\mathrm{d}x^3 + 2\,\mathrm{d}x^2\,\mathrm{d}x^4 + a(x^3, x^4)\{(\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2\} \\ &= \mathrm{d}z\,\mathrm{d}\bar{w} + \mathrm{d}w\,\mathrm{d}\bar{z} + a(w)\,\mathrm{d}w\,\mathrm{d}\bar{w}. \end{aligned} \tag{29}$$

This metric can be defined on $M = \mathbb{R}^4 = \mathbb{C}^2$, and we have a noncompact nonflat indefinite Kähler–Einstein 4-manifold. It should be noted that such a metric can be defined also on a complex torus $T = \mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2$, and the Walker 4-manifold thus constructed on the torus is nothing but the nonflat indefinite Kähler–Einstein manifold of Petean (cf. Section 3).

Case IV $(a(x^1, x^2, x^3, x^4) = -b(x^1, x^2, x^3, x^4))$. If $b(x^1, x^2, x^3, x^4) = -a(x^1, x^2, x^3, x^4)$, then all the conditions in Theorem 2 clearly holds without further conditions on a and b. Therefore, Ω_g becomes a symplectic form, and takes the simplest form as

$$\Omega_g = \mathrm{d}x^1 \wedge \mathrm{d}x^4 - \mathrm{d}x^2 \wedge \mathrm{d}x^3. \tag{30}$$

The metric takes the form

$$g = 2\,\mathrm{d}x^1\,\mathrm{d}x^3 + 2\,\mathrm{d}x^2\,\mathrm{d}x^4 + a(x^1, x^2, x^3, x^4)\{(\mathrm{d}x^3)^2 - (\mathrm{d}x^4)^2\}. \tag{31}$$

The action of the almost complex structure J in (15) reduces to

$$J\frac{\partial}{\partial x^1}=\frac{\partial}{\partial x^2},\qquad J\frac{\partial}{\partial x^2}=-\frac{\partial}{\partial x^1},\qquad J\frac{\partial}{\partial x^3}=a\frac{\partial}{\partial x^2}+\frac{\partial}{\partial x^4},$$
$$J\frac{\partial}{\partial x^4}=a\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^3},$$

which is not in general integrable for generic $a=a(x^1,x^2,x^3,x^4)$. It should be noted that if $a\ (=-b)$ depends only on (x^3,x^4) then the metric (31) is of Einstein.

7. Opposite symplectic structures in the case of $c=0$

The situation for the opposite Kähler form Ω'_g in (22) is quite different from that of the Kähler form Ω_g, as studied in the previous section.

Similarly to Theorem 2, we have the conditions for Ω'_g to be symplectic.

Theorem 6. *The opposite pseudo-Kähler form Ω'_g is a symplectic form* ($\mathrm{d}\Omega'_g=0$) *if the following partial differential equations hold*

$$H\frac{\partial a}{\partial x^1}-a\frac{\partial H}{\partial x^1}+2\frac{\partial H}{\partial x^3}=0,\qquad H\frac{\partial b}{\partial x^2}-b\frac{\partial H}{\partial x^2}+2\frac{\partial H}{\partial x^4}=0,$$
$$H\left(K\frac{\partial a}{\partial x^1}+\frac{1}{K}\frac{\partial b}{\partial x^1}\right)+\left(b\frac{\partial a}{\partial x^2}+a\frac{\partial b}{\partial x^2}\right)-2\frac{\partial a}{\partial x^4}$$
$$+H\left(a-\frac{b}{K^2}\right)\frac{\partial K}{\partial x^1}-ab\frac{\partial H}{\partial x^1}+\frac{2a}{H}\frac{\partial H}{\partial x^4}=0,$$
$$\left(b\frac{\partial a}{\partial x^1}+a\frac{\partial b}{\partial x^1}\right)+H\left(K\frac{\partial a}{\partial x^2}+\frac{1}{K}\frac{\partial b}{\partial x^2}\right)-2\frac{\partial b}{\partial x^3}$$
$$+H\left(a-\frac{b}{K^2}\right)\frac{\partial K}{\partial x^2}-ab\frac{\partial H}{\partial x^2}+\frac{2b}{H}\frac{\partial H}{\partial x^3}=0. \tag{32}$$

Here, we will not try to find the general solutions to these conditions. If H is constant, then there is a relation $b^2+4=C^4/(a^2+4)$ (C: constant), and all the conditions above reduce to the following equation:

$$\frac{\partial a}{\partial x^1}=0,\qquad \frac{\partial b}{\partial x^2}=0,$$
$$C\left(K\frac{\partial a}{\partial x^1}+\frac{1}{K}\frac{\partial b}{\partial x^1}\right)+\left(b\frac{\partial a}{\partial x^2}+a\frac{\partial b}{\partial x^2}\right)-2\frac{\partial a}{\partial x^4}+C\left(a-\frac{b}{K^2}\right)\frac{\partial K}{\partial x^1}=0,$$
$$\left(b\frac{\partial a}{\partial x^1}+a\frac{\partial b}{\partial x^1}\right)+C\left(K\frac{\partial a}{\partial x^2}+\frac{1}{K}\frac{\partial b}{\partial x^2}\right)-2\frac{\partial b}{\partial x^3}+C\left(a-\frac{b}{K^2}\right)\frac{\partial K}{\partial x^2}=0. \tag{33}$$

Since a and b must have the same arguments, we have that if $\partial a/\partial x^i = 0$ then $\partial b/\partial x^i = 0$ and vice versa. Therefore, the first two condition imply that $\partial b/\partial x^1 = 0$ and $\partial a/\partial x^2 = 0$. Then the last two condition further reduce to $\partial a/\partial x^4 = 0$ and $\partial b/\partial x^3 = 0$, which means that a and b are constant. We have the following proposition.

Proposition 7. *If H is constant, then Ω'_g is a symplectic form if and only if the Walker metric is flat.*

If H and K are both constant, then a and b must be constant, i.e., a flat case.

In Case II ($a = b$), the action of J' in (17) becomes

$$J'\frac{\partial}{\partial x^1} = \frac{1}{H}\left(-a\frac{\partial}{\partial x^2} + 2\frac{\partial}{\partial x^4}\right), \qquad J'\frac{\partial}{\partial x^2} = \frac{1}{H}\left(a\frac{\partial}{\partial x^1} - 2\frac{\partial}{\partial x^3}\right),$$
$$J'\frac{\partial}{\partial x^3} = 2\frac{\partial}{\partial x^2} + \frac{a}{H}\frac{\partial}{\partial x^4}, \; J'\frac{\partial}{\partial x^4} = -2\frac{\partial}{\partial x^1} - \frac{a}{H}\frac{\partial}{\partial x^3}. \tag{34}$$

In this case, integrability of J' cannot be expected for generic $a = a(x^1, x^2, x^3, x^4)$. The conditions in (32) are still complicated, and the symplectic Ω'_g cannot be expected too.

In Case III ($a(x^3, x^4) = b(x^3, x^4)$), even though J is integrable and Ω_g is symplectic, i.e., the Walker four-manifold admits an indefinite Kähler–Einstein structure, J' is not integrable and Ω'_g is still not symplectic. In other words, the Walker 4-manifold admits an indefinite Kähler–Einstein structure (g, J, Ω_g), and an opposite indefinite almost Hermitian–Einstein structure (g, J', Ω'_g).

Final Remark. Walker metrics appear in a natural form in the study of different kinds of geometric problems, like Osserman condition or uniqueness of the Levi-Civita connection in the pseudo-Riemannian setting [1,2]. In Case I, for all functions $a(x^3, x^4)$, $b(x^3, x^4)$ these metrics are Osserman with two-step nilpotent Jacobi operators. This means that they are Einstein self-dual or Einstein anti-self-dual (cf. [1, Theorem 4.2.5]).

More systematic survey for the generic Walker metrics with $c \neq 0$ is desirable.

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